Lesson

2-7 Inverse Variation Models

BIG IDEA Inverse and inverse square functions model many physical situations.

Inverse Variation

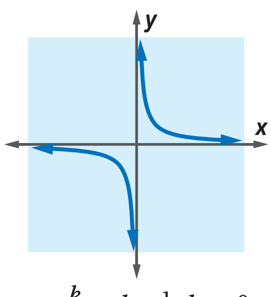
Suppose you have 6 pounds of ground meat to make into hamburger patties of equal weight. The more patties you make, the less each patty will weigh. More specifically, the weight W of each patty is related to the number N of patties produced by the equation

$$W = \frac{6}{N}.$$

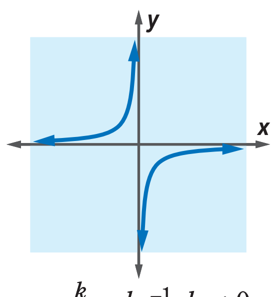
In this situation, the weight per patty *varies inversely as* (or *is inversely proportional to*) the number of patties.

STOP QY1

In general, we say that y **varies inversely as** x (or y **is inversely proportional to** x) whenever $y = \frac{k}{x}$. The parameter k is called the **constant of variation** (or **constant of proportionality**), and cannot be 0. In the hamburger situation, it is impossible for either N or W to be negative, but in other situations the domain and range may include negative values. Graphing $y = \frac{k}{x}$ for both positive and negative values of x shows that there are two basic types of graphs, depending on the value of k. Sometimes you will see the expression $\frac{k}{x}$ written as kx^{-1}.



$y = \frac{k}{x} = kx^{-1}, k > 0$



$y = \frac{k}{x} = kx^{-1}, k < 0$

Activity

Step 1 Graph $y = \frac{k}{x}$ for different positive values of k. On some graphing utilities, you can use a slider for k as shown on the next page. How does increasing the value of k affect the graph?

(continued on next page)

Vocabulary

varies inversely as, is inversely proportional to

constant of variation, constant of proportionality

inverse-square relationship

varies inversely as the square of, is inversely proportional to the square of

power function

Mental Math

Rewrite the expression as a power of a single variable, if possible.

a. $\frac{1}{x}$

b. $\frac{1}{y^2}$

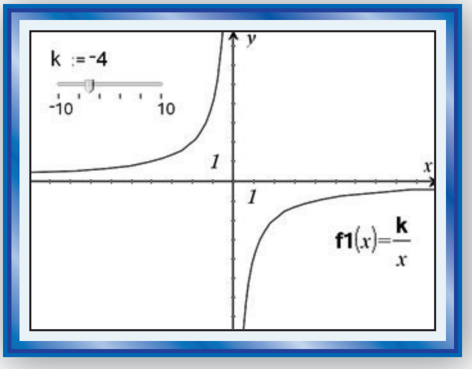
c. $(z^{-3})^5$

d. $(\sqrt{w})^{24}$

QY1

Using the relationship above, make a table of the weight W per patty for $N = 12, 24, 36,$ and 48 patties. Will the weight ever equal zero?

Step 2	Graph $y = \frac{k}{x}$ for different negative values of k. How does the sign of k affect the graph? Which features of the graph are invariant, that is, which features do not depend on the value of k?	
Step 3	Fix a particular positive value of k and pick a point on the graph with $x > 0$. Then trace the graph, moving to the left. As x gets closer to 0, what happens to the y-coordinate?	
Step 4	Now pick to a point on the graph with $x < 0$. Then trace the graph, moving to the right. As x gets closer to 0, what happens to the y-coordinate?	



The graphs suggest that there are points on the graph of $y = \frac{k}{x}$ corresponding to all real values of x except $x = 0$, and to all real values of y except $y = 0$. The domain of the function with equation $y = \frac{k}{x}$ is therefore $\{x|\ x \neq 0\}$ and the range is therefore $\{y|\ y \neq 0\}$.

The graphs in the activity are *hyperbolas*. Both hyperbolas have a *horizontal asymptote* at $y = 0$. In numerical terms, as x gets larger, y gets closer and closer to 0. In the hamburger situation, this means that with a fixed amount of meat, as the number of hamburgers increases, the weight of each burger decreases until, with enough burgers, the weight of each can be as small as you wish.

In addition, both graphs have a *vertical asymptote* at $x = 0$, because as x gets closer to 0, y gets larger and larger (or more and more negative) without ever reaching a bound.

Determining the Constant of Variation

When one quantity varies inversely as another, you can determine the constant of variation from one data point. Inverse variation is common in the physical world. For instance, according to Boyle's Law, the volume of a gas varies inversely as the pressure. Pressure is measured in kilopascals (kPa) and volume is measured in milliliters (mL).

GUIDED

Example 1

In a chemistry lab, you collect data on the pressure and volume of a gas.

Volume (mL)	20	30	40	50	60	70	80	90	100
Pressure (kPa)	253.3	160.5	120.9	101.6	84.6	70.9	64.2	53.8	49.3

a. **Find a formula relating the pressure and volume of the gas sample you studied in the lab.**

b. **Graph both the data and the model on a single set of axes.**

c. **Use residuals to assess the quality of your model.**

d. **What volume corresponds to a pressure of 40 kPa?**

Solution

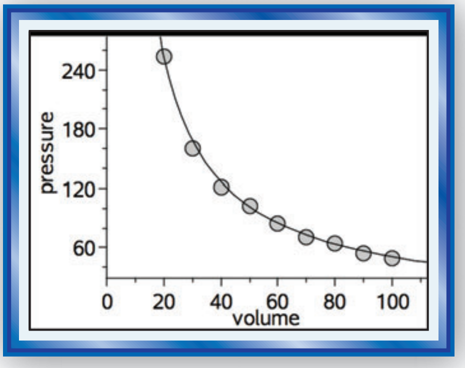
a. The model is of the form $P = \frac{k}{V}$. We start by selecting a "typical" point in the middle of the data set: $(V, P) = (60, 84.6)$.

Set $84.6 = \frac{k}{?}$ and solve for k.

$k = (\underline{\ ?\ })(\underline{\ ?\ }) = 5076.$

Therefore, $P = \underline{\ ?\ }$.

b. Enter the data into two columns of a spreadsheet. Generate a scatterplot and superimpose a graph of the model. The graph is shown at the right.



c. Add a third column that computes the predicted values and a fourth that computes the difference between the observed values and the predicted values as shown at the right.

	A vol...	B pres...	C model	D residual
◆			=5076./volu	
1	20	253.3	253.8	-0.5
2	30	160.5	169.2	-8.7
3	40	120.9	126.9	-6.
4	50	101.6	101.52	0.08
5	60	84.6	84.6	0.
6	70	70.9	72.5143	-1.61429

D1 =*b1−c1*

There are more negative than positive residuals, so we might consider $\underline{\ ?\ }$ (increasing/decreasing) the value of k we derived, but the residuals are not large and do not show a clear pattern that would suggest a different model shape. The low absolute values of the residuals and the lack of a clear pattern suggest that the model is fairly accurate.

d. Substitute 40 for P in the equation from Part a: $40 = \frac{5076}{V}$. Then solve for V: $V = \underline{\ ?\ }$ mL.

Inverse-Square Relationships

In science contexts, the relationship $y = \frac{k}{x^2} = kx^{-2}$ is very common. Such relationships are called **inverse-square relationships**, and we write y **varies inversely as the square of** x (or y **is inversely proportional to the square of** x). In many respects, inverse-square relationships are similar to inverse-variation situations.

Example 2

The force exerted by the electrical field between two charged objects is inversely proportional to the square of the distance between them. At a distance of 1.5 meters, you measure a force of 30 newtons. What will be the force at a distance of 3.0 meters? At 15 meters?

Solution First compute the constant of variation, then substitute the known distances to find the unknown force.

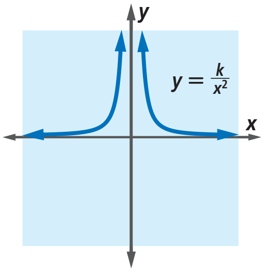
Because $F = \frac{k}{d^2}$, we can write $30\text{ N} = \frac{k}{(1.5\text{ m})^2}$ and solve for k.

$k = 30\text{ N} \cdot 1.5^2\text{ m}^2 = 67.5\text{ N} \cdot \text{m}^2$.

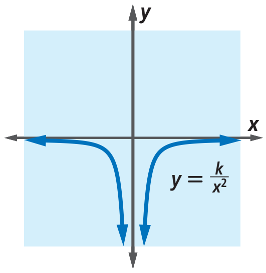
Then use the equation $F = \frac{67.5\text{ N} \cdot \text{m}^2}{d^2}$, substituting 3.0 m and 15 m for d.

When $d = 3.0$ m, the force is 7.5 N; when $d = 15$ m, the force is 0.3 N.

Graphs of $y = \frac{k}{x^2}$ share many of the features of the graphs of $y = \frac{k}{x}$.



$k > 0$



$k < 0$

Asymptotes: Both types of graphs have horizontal asymptotes at $y = 0$ and have vertical asymptotes at $x = 0$.

Domain: Because $\frac{k}{x^2}$ is defined for all values of x except when $x = 0$, the domain of $y = \frac{k}{x^2}$ is $\{x \mid x \neq 0\}$. $y = \frac{k}{x}$ has the same domain.

Range: The graphs above suggest that the range of $y = \frac{k}{x^2}$ depends on the value of k: for $k > 0$, the range is $\{y \mid y > 0\}$, while for $k < 0$, the range is $\{y \mid y < 0\}$. The inverse variation function has range $\{y \mid y \neq 0\}$. As with $y = \frac{k}{x}$, making the absolute value of k larger moves the graph of $y = \frac{k}{x^2}$ further away from the origin.

Note also that as $|x|$ increases, y gets close to zero much more quickly for $y = \frac{k}{x^2}$ than for $y = \frac{k}{x}$.

STOP QY2

▶ **QY2**

Consider $f(x) = \frac{6}{x}$ and $g(x) = \frac{6}{x^2}$. Compare $f(10)$ and $g(10)$, $f(100)$ and $g(100)$, $f(1000)$ and $g(1000)$. For each pair, which value is closer to zero?

Reciprocals of Power Functions

Recall that a **power function** is a function with an equation of the form $y = ax^n$, where n is an integer greater than 1. In this case we say that y varies directly as x^n. The reciprocal of a power function is a function of the form $y = \frac{1}{ax^n}$, or alternatively, $y = bx^{-n}$, where the coefficient $b = \frac{1}{a}$. In these cases, we say that y varies inversely as x^n. Inverse-variation functions are the reciprocals of direct-variation functions. Properties of these functions are summarized in the tables below.

Power Functions
$y = ax^n$
pass through origin
domain is $\mathbb{R}$
range is $\mathbb{R}$ (odd exponents) or $\{y \mid y \geq 0\}$ (even exponents)
rise sharply as x gets larger and larger
rise or fall sharply as x gets smaller and smaller (depending on whether n is even or odd)

Reciprocal Power Functions
$y = \frac{a}{x^n}$
have vertical asymptote at $x = 0$
domain is $\{x \mid x \neq 0\}$
range is $\{y \mid y \neq 0\}$ (odd exponents) or $\{y \mid y > 0\}$ (even exponents)
approach 0 as x gets larger and larger
approach 0 as x gets smaller and smaller

Questions

COVERING THE IDEAS

1. **Multiple Choice** In which equation does W vary inversely as the square of g?
   A $W = \frac{k}{g}$ B $W = kg^2$ C $W = \frac{k}{g^2}$ D $W = k\sqrt{g}$

2. **Multiple Choice** Which of the following is *not* a characteristic of the function $y = \frac{k}{x}$ or its graph?
   A domain is the set of all real numbers
   B horizontal asymptote at $y = 0$
   C vertical asymptote at $x = 0$
   D shape is a hyperbola

3. Suppose that y varies inversely as x, and that $y = 45$ when $x = 10$.
   a. Compute the constant of variation.
   b. Find y when $x = 2$.

4. Suppose 240 hot dogs were ordered for a picnic, and x people finished them all.
   a. Write a formula for y, the mean number of hot dogs each person ate.
   b. Graph the relation you found in Part a.
   c. Your graph has a horizontal asymptote. Find its equation, and explain its meaning in the context of the problem.

5. What kind of variation is described by $y = 11.1x^{-2}$?

6. Refer to Example 2. When two electrically-charged particles are 0.4 m apart, the force between them is 12 N. What will the force be when they are 0.2 m apart?

7. In a chemistry experiment, the data in the table below were collected on the pressure and volume of a sample of gas. According to Boyle's Law, the pressure varies inversely as the volume.

***V* (mL)**	200	220	240	260	280	300	320
***P* (kPa)**	142.5	131.2	119.6	112.9	101.7	103.2	95.4

   a. Use the data point (240, 119.6) to compute the constant of proportionality.
   b. Write a formula for P in terms of V using your value from Part a.
   c. Graph the data and the formula you found in Part b on the same set of axes.
   d. Compute the sum of squared residuals for this model.

APPLYING THE MATHEMATICS

8. True or False The time it takes you to walk a certain distance is inversely proportional to your average speed.

9. The acceleration of a falling object due to Earth's gravity varies inversely as the square of the object's distance from the center of Earth.

a. Earth's radius at sea level is about 6378 km and the acceleration due to gravity on Earth's surface is about 9.8 $\frac{\text{m}}{\text{s}^2}$. Compute the constant of proportionality.

b. Compute the acceleration due to Earth's gravity for an object in orbit 10,000 km above Earth's surface.

c. Compute the acceleration due to Earth's gravity for an object as far away as the Moon, 384,400 km from Earth's center.

REVIEW

10. Data records show the distance a ball traveled when hit at various angles at a constant bat velocity of 100 mph. **(Lesson 2-6)**

Angle (degrees)	35	40	55	65	70	75
Distance (reached feet)	294	308	294	239	201	156

a. Construct a scatterplot for these data.

b. Find a quadratic model for these data.

11. A wrecking ball swings to knock down a building. The following data were collected giving the height of the ball during its swing at various distances from the building. **(Lesson 2-6)**

Distance (feet)	100	75	50	25	0	–25
Height reached (feet)	601	536	499	501	510	554

a. Find a quadratic regression model for these data.

b. Determine the sum of squared residuals for your model.

12. Consider the function $r: t \rightarrow 5(0.83)^t$. **(Lesson 2-4)**

a. Give the domain of r.

b. Give the range of r.

c. State equations for any asymptotes of the graph of r.

13. Henry looked at the census data for Iowa City, Iowa, for the years 1960, 1970, 1980, 1990 and 2000 and used it to create a linear model of population growth. You do not need to have the data to answer these questions. **(Lesson 2-2)**

a. Henry used his model to predict the population of Iowa City in 1984. Was his prediction a result of interpolation or extrapolation?

b. Henry also used his model to predict the population of Iowa City in 2016. Would you expect his prediction for 1984 or his prediction for 2016 to have a larger error? Explain your answer.

14. The table at the right shows the winning jumps in the men's long jump event at the Olympic games. (**Lesson 2-3**)

a. Make a scatterplot of these data.

b. Find the line of best fit predicting the winning jump for a given year.

c. What does the slope of the line tell you about the average rate of change in the length of the winning long jump from 1896 to 2004?

d. Use the line of best fit to predict the winning jump for the Beijing Olympics in 2008.

e. What is the residual of the prediction? (The actual jump by Irving Jahir Saladino Aranda was 8.34 m.)

f. Which data points seem to be outliers here? Give a plausible explanation for why those outliers might have occurred.

Dwight Phillips (right) with John Moffitt, silver medalist

Year	Gold Medalist	Jump
1896	Ellery Clark, United States	6.34 m
1900	Alvin Kraenzlein, United States	7.19 m
1904	Myer Prinstein, United States	7.34 m
1908	Francis Irons, United States	7.48 m
1912	Albert Gutterson, United States	7.60 m
1920	William Pettersson, Sweden	7.15 m
1924	DeHart Hubbard, United States	7.45 m
1928	Edward B. Hamm, United States	7.74 m
1932	Edward Gordon, United States	7.64 m
1936	Jesse Owens, United States	8.06 m
1948	Willie Steele, United States	7.82 m
1952	Jerome Biffle, United States	7.57 m
1956	Gregory Bell, United States	7.83 m
1960	Ralph Boston, United States	8.12 m
1964	Lynn Davies, Great Britain	8.07 m
1968	Robert Beamon, United States	8.90 m
1972	Randy Williams, United States	8.24 m
1976	Arnie Robinson, United States	8.35 m
1980	Lutz Dombrowski, East Germany	8.54 m
1984	Carl Lewis, United States	8.54 m
1988	Carl Lewis, United States	8.72 m
1992	Carl Lewis, United States	8.67 m
1996	Carl Lewis, United States	8.50 m
2000	Ivan Pedroso, Cuba	8.55 m
2004	Dwight Phillips, United States	8.59 m

Source: Athletics Weekly 2008

15. a. Which is generally least affected by outliers in the data set, the mean or median?

b. If a data set is skewed with a tail to the left, then which is larger: the mean or median? (**Lessons 1-3, 1-2**)

EXPLORATION

16. According to the mathematics in the lesson, it is never possible to entirely escape Earth's gravity: since gravitational force varies inversely as the square of the distance, no matter how large the distance gets, the force of gravity from Earth never actually equals zero. Yet astronomical missions such as the Voyager continue to travel away from Earth, without burning rockets continually. Research the concept of escape velocity and explain why such missions are possible.

17. In Question 7, you used a data point to determine an inverse-variation model for the data. Experiment with different data points to see if you can find a model with a lower sum of squared residuals.

QY ANSWERS

1. no;

N	12	24	36	48
W	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{8}$

2. $g(x)$ is closer to zero than $f(x)$ for $x = 10, 100,$ and 1000.